Classification of Group Actions on Kirchberg Algebras

by

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Introduction

Classification results of group actions on the injective factor of type II_1:

Theorem

$\exists^1$ Cocycle conjugacy class of outer actions of

- $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$: (Connes 1975),
- Finite groups: (Jones 1980),
- Amenable discrete groups: (Ocneanu 1985).

Discrete amenable group actions on the other injective factors are also classified:
(竹崎 Takesaki, Sutherland, 河東 Kawahigashi , 片山 Katayama).

What about the case of nuclear C^*-algebras?

There is a hope to classify outer actions of "reasonable" amenable groups on Kirchberg algebras.

<u>Plan of the lectures</u>

I. Overview.

II. The classification of $\mathbb{Z}$-actions (after 中村 Nakamura).

III. Poly-$\mathbb{Z}$ group actions.
Work in progress with 松井宏樹 Hiroki Matui.

Kirchberg Algebras

Definition
A Kirchberg algebra is a purely infinite, simple, nuclear, separable C^*-algebra.

Example
Cuntz algebra $\mathcal{O}_n = C^*\{S_1, S_2, \cdots, S_n\}$:

$$S_i^* S_j = \delta_{ij} 1,$$

$$\sum_{i=1}^{n} S_i S_i^* = 1, \quad \text{if } n < \infty.$$

Theorem (Kirchberg, Phillips)
Kirchberg algebras are completely classified by KK-theory.

Definition

$\rho_1, \rho_2 \in \mathrm{Hom}(A, B)$ are asymptotically unitarily equivalent, $\rho_1 \overset{\text{as.u.}}{\sim} \rho_2$,
$\overset{\text{def}}{\Leftrightarrow}$ $\exists \{u(t)\}_{t \geq 0}$ continuous path in $U(B)$ s.t.

$$\lim_{t \to \infty} \|\mathrm{Ad}\, u(t) \circ \rho_2(x) - \rho_1(x)\| = 0, \ \forall x \in A.$$

$\widehat{H}(A, B)$:=The set of unital homomorphisms from A to B modulo asym. u. equivalence.

Theorem (Kirchberg, Phillips)
A, B: Unital Kirchberg algebras with
$[1_A]_0 = 0$ in $K_0(A)$, and $[1_B]_0 = 0$ in $K_0(B)$.
Then

$$\widehat{H}(A, B) \cong KK(A, B).$$

The main ingredient of the proof:

Theorem (Kirchberg 1994)

(1) $\mathcal{O}_\infty$-theorem:
For any Kirchberg algebra A,

$$A \otimes \mathcal{O}_\infty \cong A.$$

cf. $\mathcal{O}_\infty \overset{\mathrm{KK}}{\sim} \mathbb{C}$.

(2) $\mathcal{O}_2$-theorem:
For any separable nuclear simple unital A,

$$A \otimes \mathcal{O}_2 \cong \mathcal{O}_2.$$

cf. $\mathcal{O}_2 \overset{\mathrm{KK}}{\sim} \{0\}$.

$\mathbb{Z}$-actions

$\alpha \in \mathrm{Aut}(A)$ is said to be aperiodic if α^n is outer for all $n \in \mathbb{Z} \setminus \{0\}$.

Theorem (Nakamura 1999)
A: Kirchberg algebra
For aperiodic $\alpha, \beta \in \mathrm{Aut}(A)$, T.F.A.E.

(1) $KK(\alpha) = KK(\beta)$,
(2) $\exists \gamma \in \mathrm{Aut}(A)$, $\exists u \in U(A)$ s.t.
$KK(\gamma) = 1$, and $\gamma^{-1} \circ \alpha \circ \gamma = \mathrm{Ad} u \circ \beta$.

The main ingredient of the proof:

Lemma
Any aperiodic automorphism of a Kirchberg algebra has the Rohlin property.

Definition

$\alpha \in \mathrm{Aut}(A)$ has the Rohlin property

$\overset{\text{def}}{\Leftrightarrow}$ $\forall \varepsilon > 0$, $\forall k \in \mathbb{N}$, $\forall F \subset A$ finite,

$\exists \{e_i\}_{i=0}^{k-1} \cup \{f\}_{i=0}^{k} \subset P(A)$ s.t.

$$\sum_{i=0}^{k-1} e_i + \sum_{i=0}^{k} f_i = I,$$

$$\|\alpha(e_i) - e_{i+1}\| < \varepsilon, \quad i = 0, 1, \cdots, k-2,$$

$$\|\alpha(f_i) - f_{i+1}\| < \varepsilon, \quad i = 0, 1, \cdots, k-1,$$

$$\|e_i x - x e_i\| < \varepsilon, \quad x \in F,\ i = 0, 1, \cdots, k-1,$$

$$\|f_i x - x f_i\| < \varepsilon, \quad x \in F,\ i = 0, 1, \cdots, k.$$

Equivalence Relations

G: Discrete group,
A, B: Unital C^*-algebras,
$\alpha : G \to \mathrm{Aut}(A)$, $\beta : G \to \mathrm{Aut}(B)$: Actions.

• α and β are conjugate, $\alpha \cong \beta$
$\overset{\text{def}}{\Leftrightarrow}$ $\exists \theta : A \to B$ isomorphism s.t. $\theta^{-1} \circ \beta_g \circ \theta = \alpha_g$.

• α and β are outer conjugate
$\overset{\text{def}}{\Leftrightarrow}$ $\exists \theta : A \to B$ isomorphism, $\exists u_g \in U(A)$ s.t. $\theta^{-1} \circ \beta_g \circ \theta = \mathrm{Ad} u_g \circ \alpha_g$.

• α and β are cocycle conjugate
$\overset{\text{def}}{\Leftrightarrow}$ $\exists \theta : A \to B$ isomorphism, $\exists u_g \in U(A)$ s.t. $\theta^{-1} \circ \beta_g \circ \theta = \mathrm{Ad} u_g \circ \alpha_g$ and
$u_g \alpha_g(u_h) = u_{gh}$ (1-cocycle relation).

If moreover $A = B$ and $KK(\theta) = 1$, we say that α and β are KK-trivially cocycle conjugate.

Finite Group Actions

Finite group actions are difficult to deal with.

For instance, though $\mathcal{O}_2 \overset{KK}{\sim} \{0\}$,

Theorem (I. 2004)
$\forall G_0, G_1$: Uniquely 2-divisible countable abelian groups,
$\exists \alpha$: Outer $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{O}_2$ s.t.

$$K_i(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2) = G_i, \quad i = 0, 1.$$

Moreover, $\exists \{u(t)\}_{t \geq 0} \subset U(\mathcal{O}_2)$ continuous path s.t.

$$u(t)^2 = 1, \quad \alpha(u(t)) = u(t),$$

$$\alpha(x) = \lim_{t \to \infty} \mathrm{Ad}u(t)(x), \quad \forall x \in \mathcal{O}_2.$$

Note that $K_*(\mathcal{O}_2 \rtimes_{\mathrm{Ad}u(t)} \mathbb{Z}_2) = \{0\}$.

Theorem (I. 2004)
A : Kirchberg algebra satisfying UCT.
G : Finite group.
α, β : G-actions on A with the Rohlin property.
Then T.F.A.E.

(1) $\forall g \in G,\ K_*(\alpha_g) = K_*(\beta_g)$.
(2) $\exists \theta \in \mathrm{Aut}(A)$, s.t. $K_*(\theta) = 1$ and

$$\theta^{-1} \circ \alpha_g \circ \theta = \beta_g.$$

Theorem (I. 2004)
A : Unital simple separable nuclear,
G : Finite groups,
α : Outer action of G.
Then

$$(A \otimes \mathcal{O}_2, \alpha \otimes \mathrm{id}) \cong (\mathcal{O}_2, \mu),$$

where μ is a unique action with the Rohlin property,
e.g. $\mu(S_1) = S_1$, $\mu(S_2) = -S_2$.

$\{S_i\}_{i=1}^{\infty}$: Canonical generators of $\mathcal{O}_\infty$.
$\mathcal{H} := \overline{\text{Span}\{S_i\}_{i=1}^{\infty}}$.

An action μ of G on $\mathcal{O}_\infty$ is quasi-free
$\overset{\text{def}}{\Leftrightarrow}$ $\forall g \in G,\ \mu_g(\mathcal{H}) = \mathcal{H}$.

Theorem (I.)
A : Kirchberg algebra,
G : Finite group,
α : Outer action of G on A.
Then for any sequence of quasi-free actions $\{\mu^{(i)}\}_{i=1}^{\infty}$,

$$(A, \alpha) \cong (A \otimes \bigotimes_{i=1}^{\infty} \mathcal{O}_\infty, \alpha \otimes \bigotimes_{i=1}^{\infty} \mu^{(i)}).$$

Conjecture
The statement is still true for any discrete amenable group (if conjugacy is replaced by cocycle conjugacy).

The conjecture is true for poly-$\mathbb{Z}$ groups.

Poly-$\mathbb{Z}$ groups

Definition

A discrete group G is a poly-$\mathbb{Z}$ group $\overset{\text{def}}{\Leftrightarrow}$ if there exists a normal series

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_m = G,$$

such that

$$G_{i+1}/G_i \cong \mathbb{Z}^{n_i}.$$

The number

$$h(G) = \sum_{i=0}^{m-1} n_i$$

is said to be the Hirsch length of G.

Example

If $h(G) = 2$, either $G = \mathbb{Z}^2$ or

$$G =< a, b | aba^{-1}b = e >= \pi_1(\text{Klein bottle}).$$

Definition

A: Unital C^*-algebra.

A is strongly self-absorbing
$\overset{\text{def}}{\Leftrightarrow}$ $\exists \rho : A \to A \otimes A$ isomorphism s.t. ρ and $A \ni x \mapsto x \otimes 1 \in A \otimes A$ is approximately unitarily equivalent.

Example

$\mathcal{O}_2$, $\mathcal{O}_\infty$, $\mathcal{O}_\infty \otimes B$, where B is a UHF algebra with $B \cong B \otimes B$.

These exhaust all the strongly self-absorbing Kirchberg algebras satisfying UCT.

Theorem (Dadarlat-Winter)

A : K_1-injective and strongly self-absorbing.

The homotopy group $\pi_n(\mathrm{Aut}(A))$ is trivial for any n.

Theorem (Matui-I.)
A: Strongly self-absorbing Kirchberg algebra,
G: Poly-$\mathbb{Z}$ group.
Then there exists a unique cocycle conjugacy class of outer G-actions on A.

Theorem (Matui-I.)
A: Kirchberg algebra,
G: Poly-$\mathbb{Z}$ group,
α: Outer G-action on A,
μ: Outer G-action on $\mathcal{O}_\infty$.
Then α is cocycle conjugate to $\alpha\otimes\mu$ on $A\otimes\mathcal{O}_\infty$.

Corollary
Any outer action of $\mathbb{Z}^n$ on a Kirchberg algebra has the Rohlin property.

Higher invariant

A: Kirchberg algebra,
G: Discrete group,
α, β: G-actions on A.

The homomorphism

$$G \ni g \mapsto KK(\alpha_g) \in KK(A, A)^{-1}$$

is an invariant for KK-trivial cocycle conjugacy.

When $KK(\alpha_g) = KK(\beta_g)$,
is there any obstruction for KK-trivial cocycle conjugacy?

Since $\alpha_g \otimes \mathrm{id}_{\mathbb{K}} \overset{\mathrm{h}}{\sim} \beta_g \otimes \mathrm{id}_{\mathbb{K}}$ in $\mathrm{Aut}(A \otimes \mathbb{K})$,
one can take a homotopy $\{\sigma_g^{(t)}\}_{t \in [0,1]}$.

$\omega(g,h) :=$ The class of the loop

$$[0,1] \ni t \mapsto \sigma_g^{(t)} \circ \sigma_h^{(t)} \circ \sigma_{gh}^{(t)^{-1}} \in \mathrm{Aut}(A \otimes \mathbb{K})_0$$

in $\pi_1(\mathrm{Aut}(A \otimes \mathbb{K})_0)$.

ω satisfies the 2-cocycle relation:

$$g \cdot \omega(h,k) - \omega(gh,k) + \omega(g,hk) - \omega(g,k) = 0.$$

The cohomology class $c(\alpha,\beta)$ of ω in $H^2(G, \pi_1(\mathrm{Aut}(A \otimes \mathbb{K}))_0)$ is an obstruction for KK-trivial cocycle conjugacy.

Theorem (Dadarlat)
$\pi_1(\mathrm{Aut}(A \otimes \mathbb{K})_0) \cong KK^1(A,A)$.

$c(\alpha,\beta) \in H^2(G, KK^1(A,A))$.

Classification

Theorem (Matui-I.)
A: Kirchberg algebra,
$G = \mathbb{Z}^2$ or $G = < a, b | aba^{-1}b = e >$,
α, β: Outer G-actions of G on A with
$KK(\alpha_g) = KK(\beta_g)$.
Then T.F.A.E.
(1) α and β are KK-trivially cocycle conjugate,
(2) $c(\alpha, \beta) = 0$.

For a poly-$\mathbb{Z}$ group G, $h(G)$ coincides with the cohomological dimension of G, that is,

$$H^n(G, M) = \{0\}$$

for $\forall n > h(G)$ and $\forall$ G-module M.

For G with $h(G) > 2$, the two conditions $KK(\alpha_g) = KK(\beta_g)$ and $c(\alpha, \beta) = 0$ are not enough for KK-trivial cocycle conjugacy.